Making all equalities equal

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Abstract

1 Equalities

There are multiple ways of defining equalities in a theorem prover. In the next sections, they will be defined.

1.1 Imports

First, it will be necessary to give some agda arguments:

```
{-# OPTIONS --cubical --cumulativity #-}
module paper where
```

The cubical flag is necessary because we are using cubical equality, and the cumulative flag is also necessary for level subtyping,

```
open import Agda.Primitive.Cubical using (I; i0; i1)
```

Private variables are required, so it is not necessary to redefine them later as an implicit variable.

```
open import Cubical.Core.Primitives using (Level; ℓ-max)

private variable
  ℓ ℓ' : Level
  A : Set ℓ
```

This library loads Cubical Agda Primitives as the equality interval.

1.2 Martin-Löf Equality

At the beginning of Agda and in most theorems proves, equality is given by Martin-Löf's definition:

```
module Martin-Löf {A : Set ℓ} where
  data _≡_ (x : A) : A → Set ℓ where
    refl : x ≡ x
```

This equality is very convenient in proof assistances like Agda because it is possible to pattern match using them:

```
private variable
  x y z : A

sym : x ≡ y → y ≡ x
sym refl = refl

trans : x ≡ y → y ≡ z → x ≡ z
trans refl refl = refl
```

But the problem of this equality is that it does not handle extensionality and other axioms very well.

```
module FunExt {A : Set ℓ} {B : Set ℓ'} where
  open Martin-Löf

  funExt-Type = {f g : A → B}
    → ((x : A) → f x ≡ g x) → f ≡ g
```

1.3 Cubical Equality

To solve this problem, Agda adopted cubical type theory that equality is a function from the path to type:

```
module CubicalEquality {A : Set ℓ} where
  postulate
    PathP : (A : I → Set ℓ) → A i0 → A i1 → Set ℓ

  _≡_ : A → A → Set ℓ
  _≡_ = PathP λ _ → A
```

From this equality, I will define reflection, symmetry and extensionality:

```
module CubicalResults {A : Set ℓ} {B : Set ℓ'} where
  open import Cubical.Core.Primitives

  private variable
    x y z : A

  refl : x ≡ x
  refl {x = x} = λ _ → x
```

```
sym : x ≡ y → y ≡ x
sym p i = p (~ i)

funExt : {f g : A → B}
  → ((x : A) → f x ≡ g x) → f ≡ g
funExt p i x = p x i
```

The operator ~ invert the interval. If the interval i goes from *i0* to *i1*, the interval ~ i goes from *i1* to *i0*.

1.4 Leibniz equality

Leibniz equality is defined in this way: If a is equal to b, then for every propositional P, if $P\ a$, then $P\ b$. The main idea is that if both values are equal, then they are seen equal for every angle.

```
module LeibnizEquality {A : Set} where
  _≐_ : A → A → Set₁
  a ≐ b = (P : A → Set) → P a → P b
```

2 Joining all equalities

All equalities have something in common. They are all equal to each other. So it will be defined as a common record that all equalities should have. In the next definition, all equalities are equal to cubical equality:

```
open import Cubical.Foundations.Prelude
open import Cubical.Foundations.Isomorphism
open import Cubical.Foundations.Equiv
open import Cubical.Foundations.Univalence
open import Cubical.Foundations.Function
open import Cubical.Data.Equality

module _ {ℓ} {A : Set ℓ'} where
  ≜-Type = A → A → Set ℓ
  private
    ℓ₁ = ℓ-max ℓ' ℓ

  private variable
    x y z : A

  record IsEquality (_≜_ : ≜-Type) : Set (ℓ-suc ℓ₁) where
    constructor eq
    field
       ≜-≡-≡ : let
```

```
      x≡y : Type ℓ ₁
      x≡y = x ≡ y

      x≙y : Type ℓ ₁
      x≙y = x ≙ y

      in _≡_ {ℓ-suc ℓ ₁} x≙y x≡y

  ≡-≡-≙ : let
    x≡y : Type ℓ ₁
    x≡y = x ≡ y
    x≙y = x ≙ y
    in x≡y ≡ x≙y
  ≡-≡-≙ = sym ≙-≡-≡

module _ {_≙_ : ≙-Type} where
  sym-Equality : (≡-≡-≙ : {x y : A} → let
    x≡y : Type ℓ ₁
    x≡y = x ≡ y
    x≙y = x ≙ y
    in x≡y ≡ x≙y)
    → IsEquality _≙_
  sym-Equality ≡-≡-≙ = eq (sym ≡-≡-≙)

record Equality : Set (ℓ-suc ℓ ₁) where
  constructor eqC
  field
    _≙_ : ≙-Type
    { isEquality } : IsEquality _≙_

EqFromInstance : {≙ : ≙-Type} → IsEquality ≙ → Equality
EqFromInstance inst = eqC _ { inst }

eqsEqual : (_≙ ₁_ _≙ ₂_ : ≙-Type)
  { ≙ ₁-eq : IsEquality _≙ ₁_ }
  { ≙ ₂-eq : IsEquality _≙ ₂_ }
  → ∀ {x y} → let
    x≙ ₁y : Type ℓ ₁
    x≙ ₁y = x ≙ ₁ y

    x≙ ₂y : Type ℓ ₁
    x≙ ₂y = x ≙ ₂ y

    in _≡_ {ℓ-suc ℓ ₁} x≙ ₁y x≙ ₂y
eqsEqual _ _ { eq ≙-≡-≡ ₁ } { eq ≙-≡-≡ ₂ } = ≙-≡-≡ ₁ • sym ≙-≡-≡ ₂
```

It will be defined for each equality, its instance:

2.1 Cubical Equality

The simplest example is cubical equality hence this equality is already equal in itself.

```
module _ {a} {A : Set a} where
  instance
    ≡-IsEquality : IsEquality {A = A} _≡_
    ≡-IsEquality = eq refl
  ≡-Equality : Equality {ℓ = a}
  ≡-Equality = eqC _≡_
```

2.2 Martin-Löf equality

The proof of Martin-Löf equality is more difficult, but it is already in the Cubical library as p-c.

```
instance
  ≡p-IsEquality : IsEquality {A = A} _≡p_
  ≡p-IsEquality = sym-Equality p-c
≡p-Equality : Equality {ℓ = a}
≡p-Equality = eqC _≡p_
```

2.3 Isomorphism

Isomorphism is equality between types.

```
module _ {ℓ} where
  univalencePath' : {A B : Type ℓ} → (A ≡ B) ≡ (A ≃ B)
  univalencePath' {A} {B} =
    ua {ℓ-suc ℓ} {A ≡ B} {A ≃ B} (compEquiv (univalence {ℓ} {A} {B})
    (isoToEquiv (iso {ℓ} {ℓ-suc ℓ}
    (λ x → x) (λ x → x) (λ b i → b) λ a i → a)))
```

univalencePath is already defined in Agda library, but with $A \simeq B$ instead of *Lifted* $(A \simeq B)$. This change can be done because of the cumulative flag.

```
instance
  ≃-IsEquality : IsEquality
    {A = Type ℓ} _≃_
  ≃-IsEquality = sym-Equality univalencePath'
≃-Equality : Equality {ℓ = ℓ}
≃-Equality = eqC _≃_
```

2.4 Leibniz Equality

The hardest equality to prove that is equality is the Leibniz Equality.

```
liftIso : {A : Type ℓ} {B : Type ℓ'}
  → Iso {ℓ} {ℓ'} A B → Iso {ℓ-max ℓ ℓ'} {ℓ-max ℓ ℓ'} A B
liftIso f = iso fun inv
  (λ x i → rightInv x i) (λ x i → leftInv x i)
```

This liftIso will be used to lift the Isomorphism to types of the same maximum level of both.

```
where open Iso f

open import leibniz
open Leibniz
```

It is importing the definition of Leibniz equality made by [**?**]. In this work, there is already proof of the isomorphism between Leibniz and Martin-Löf equality.

```
module FinalEquality {A : Set} where
  open MainResult A
  private variable
    x y z : A

  ≐≅≡ : Iso (x ≐ y) (x ≡p y)
  ≐≅≡ = iso j i (ptoc ∘ ji) (ptoc ∘ ij)
```

In Cubical Library, the definition of isomorphism uses cubical equality instead of Martin-Löf equality when we have to prove that ∀ *x* → *from (to x)* ≡ *x* and ∀ *x* → *to (from x)* ≡ *x*. ptoc is necessary to do this conversion from these equalities.

```
≐≡≡ : (x ≐ y) ≡c (x ≡p y)
≐≡≡ = let lifted = liftIso ≐≅≡ in isoToPath lifted
```

Using the univalence and liftIso defined previously, it is possible to transform the isomorphism into equality.

```
open IsEquality

instance
  ≐-IsEquality : IsEquality {A = A} _≐_
  ≐-IsEquality = eq λ {x} {y} → ≐≡≡ •
    λ i → ≡p-IsEquality {ℓ-zero} .≙-≡-≡ {x} {y} i

≐-Equality : Equality {ℓ = ℓ-suc ℓ-zero}
≐-Equality = eqC _≐_
```

The last pass is to join the three equalities between equalities: Leibniz to Martin-Löf to cubical equality.

3 New Equalities types

The equalities used previously were defined using cubical equality. Now I will define them using other equalities.

```
module Equalities {a ℓ} {A : Set a} where
  private
    ≜-Type' = ≜-Type {a} {ℓ} {A}
    ℓ 1 = ℓ-max a ℓ

  private variable
    x y z : A
```

Loaded the modules using the levels to be more generic.

```
module _
  (Eq 1 : Equality {_} {ℓ} {A})
  where

  open Equality Eq 1 renaming (_≜_ to _≡ 1 _; isEquality to eq 1)
```

I am importing generic equality to use it to define more generic equality.

```
record IsEquality 2 (_≜_ : ≜-Type') : Set (ℓ-suc ℓ 1) where
  constructor eq
  field
    ≜-≡-≡ : let
      x≡y : Type ℓ 1
      x≡y = x ≡ 1 y
```

Different from previously definition of IsEquality, the cubical equality defined in the line above was substituted by the more generic equality $\equiv_1$.

```
x≜y : Type ℓ 1
x≜y = x ≜ y

in _≡_ {ℓ-suc ℓ 1} x≜y x≡y

≡-≡-≜ : let
  x≡y : Type ℓ 1
  x≡y = x ≡ 1 y
  x≜y = x ≜ y
  in x≡y ≡ x≜y
≡-≡-≜ = sym ≜-≡-≡
```

The rest of the definition is the same.

```
instance
  ≜-isEquality : IsEquality _≜_
  ≜-isEquality = eq (≜-≡-≡ • IsEquality.≜-≡-≡ eq 1)
```

From a more generic definition of equality, it is easily possible to return to the less generic definition.

```
module _
  (Eq 2 : Equality {_} {ℓ} {A})
  where

  open Equality Eq 2 renaming (_≜_ to _≡ 2_; isEquality to eq 2)
```

I am defining a new generic equality to prove that it is an equality of type 2:

```
eqsEqual 2 : let
  x≜ 1y : Type ℓ 1
  x≜ 1y = x ≡ 1 y

  in x≜ 1y ≡ (x ≡ 2 y)
eqsEqual 2 = eqsEqual _≡ 1_ _≡ 2_

instance
  ≡ 2-Equality 2 : IsEquality 2 _≡ 2_
  ≡ 2-Equality 2 = eq (sym eqsEqual 2)
    where open IsEquality

module _ {_≜_ : ≜-Type} where
  sym-Equality 2 : (≡-≡-≜ : {x y : A} → let
    x≡y : Type ℓ 1
    x≡y = x ≡ 1 y
    in x≡y ≡ (x ≜ y))
    → IsEquality 2 _≜_
  sym-Equality 2 ≡-≡-≜ = eq (sym ≡-≡-≜)
```

Given a symmetric definition of the previous equality, it is easy to prove that it is also equality of type 2.

3.1 Everything is an equality

In this part, a relation is equality when it is equal (using general equality) to cubical equality.

```
module _
  (Eq 3 : Equality {A = Set ℓ 1})
  where
```

```
open Equality Eq 3 renaming (_≜_ to _≡ 3_; isEquality to eq 3)

record IsEquality 3 (_≜_ : ≜-Type') : Set (ℓ-suc ℓ 1) where
  constructor eq
  field
    ≜-≡-≡ : (x ≜ y) ≡ 3 (x ≡ 1 y)
```

With this definition of equality, it is possible to prove that if equality is equal to cubical equality, so it is equal (using the general or cubical equality) to the cubical equality.

```
instance
  ≜-isEquality 2 : IsEquality 2 _≜_
  ≜-isEquality 2 = eq (transport (IsEquality.≜-≡-≡ eq 3) ≜-≡-≡)

  ≜-isEquality : IsEquality _≜_
  ≜-isEquality = eq (IsEquality 2.≜-≡-≡ ≜-isEquality 2 • IsEquality.≜-≡-≡ eq 1)
```

This is proof that the symmetric definition of equality is also valid.

```
≡-≡-≜ : (x ≡ 1 y) ≡ 3 (x ≜ y)
≡-≡-≜ = let
  α 1 = IsEquality.≡-≡-≜ eq 3
  α 2 = IsEquality.≜-≡-≡ eq 1
  α 3 = IsEquality.≡-≡-≜ ≜-isEquality
  in transport α 1 (α 2 • α 3)
```

It is possible to prove that a general equality is equality from this definition:

```
module _
  (Eq 2 : Equality {_} {ℓ} {A})
  where

  open Equality Eq 2 renaming (_≜_ to _≡ 2_; isEquality to eq 2)

  instance
    ≡ 2-Equality 3 : IsEquality 3 _≡ 2_
    ≡ 2-Equality 3 = eq α
      where
        open IsEquality eq 3
        α : (x ≡ 2 y) ≡ 3 (x ≡ 1 y)
        α = transport ≡-≡-≜ (IsEquality 2.≜-≡-≡ (≡ 2-Equality 2 (eqC _≡ 2_)))
```

If there is proof of symmetrical equality, so it is also equality from this definition:

```
module _ {_≜_ : ≜-Type'} where
```

```
sym-Equality₃ :
  (≡-≡-≜ : ∀ {x y} → (x ≡₁ y) ≡₃ (x ≜ y))
  → IsEquality₃ _≜_
sym-Equality₃ ≡-≡-≜ = eq (let
  α₁ = IsEquality.≡-≡-≜ eq₃
  α₂ = transport (sym α₁) ≡-≡-≜
  in transport α₁ (sym α₂))
```

4 Using the definitions

The best part of defining all of this stuff is that it is now easy to prove that Leibniz equality is equality.

```
module LeibnizFromPEquality {A : Set} where
  open Equalities {ℓ-zero} {ℓ-suc ℓ-zero}

  _≡p₁_ : A → A → Set₁
  x ≡p₁ y = x ≡p y
```

I redefined this equality because it must be a set of universe one. And because of that, I have to prove again that this is an equality:

```
instance
  ≡p₁-isEquality : IsEquality _≡p₁_
  ≡p₁-isEquality = eq λ {x y} → (sym λ i → let
    α : Type₁
    α = p-c {ℓ-zero} {x = x} {y = y} i
    in α)
```

With just one line of code, it is possible now to prove that Leibniz equality is an equality from Martin-Löf Equality.

```
leibniz : IsEquality {A = A} _≐_
leibniz = IsEquality₂.≜-isEquality {Eq₁ = eqC _≡p_} (eq FinalEquality.≐≡≡)
```